

Characterization in bi-parameter space of a non-ideal oscillator

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Abstract

We investigate the dynamical behavior of a non-ideal Duffing oscillator, a system composed of a mass-spring-pendulum driven by a DC motor with limited power supply. To identify new features on Duffing oscillator parameter space due to the limited power supply, we provide an extensive numerical characterization in the bi-parameter space by using Lyapunov exponents. Following this procedure, we identify remarkable new periodic windows, the ones known as Arnold tongues and also shrimp-shaped structures. Such windows appear in highly organized distribution with typical self-similar structures for the shrimps, and, surprisingly, codimension-2 bifurcation as a point of accumulations for the tongues.

Keywords: Chaos, Arnold tongues, Shrimps, Coupled oscillators

1. Introduction

In recent years, there has been an increasing amount of work on nonlinear dynamics characterizing the possible structures in two-dimensional control parameter (bi-parameter) space [1]. Accordingly, periodic windows with important features, mainly shrimp-shaped structures [2] and Arnold tongues [3, 4, 5], have been identified in several systems such as two-gene model [6], impact oscillator [7, 8], dissipative model of relativistic particles [9], tumor growth model [10], Chua's circuit [11, 12, 13], prey-predator model [14], and Red Grouse population model [15].

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In the nonlinear dynamics context, oscillators with mechanical coupling have recently attracted a significant attention due to the complexity of the dynamics for high degree-of-freedom devices and possible applications to advanced technologies [16, 17, 18, 19, 20]. Among the class of mechanical coupling oscillators an interesting example is the mass-spring-pendulum system [21, 22]. Svoboda and collaborators studied a system of masses with a pendulum, where the pendulum is attached to one mass of a chain of masses connected by springs [23]. They showed that auto parametric resonance can arise. In Reference [24] was investigated the influence of nonlinear spring on the auto parametric system. It was verified the existence of regular and chaotic motions.

In this work, we investigate the parameter space organization of a non-ideal Duffing oscillator, namely, the mass-spring-pendulum system. Duffing oscillator is a forced oscillator with a nonlinear elasticity, and it is described by a non linear differential equation of second-order that has been used in a variety of physical processes. This oscillator is well known in engineering science, and it has been used to model the dynamics of types of electrical and mechanical systems. Among and collaborators experimentally studied signal amplification in a nanomechanical Duffing resonator via stochastic resonance [25]. The Duffing oscillator is also a useful model to study the dynamics behavior of structural systems, such as columns, gyroscopes, and bridges [26].

The non-ideal character of the studied oscillator is a consequence of the fact that the source of energy is given by a DC motor with limited power supply [27, 28]. Previous studies of this system have shown a rich dynamical behavior with several nonlinear phenomena, like quasi-periodic attractors, chaotic regimes, crises, coexistence of attractors, and fractal basin boundaries [29, 30, 31]. Here, our main purpose is to provide a global parameter analysis of the behavior of this oscillator with a mechanical coupling. The main features found in the parameter space were the self-similar structures, and codimension-2 bifurcation as a point of accumulations for the Arnold tongues. Comparing with results from parameter spaces of ideal oscillators, these Arnold tongue attributes are a consequence of the non-ideal character of this oscillator.

This paper is organized as follows. In Section 2 we present the mathematical description of the non-ideal Duffing oscillator. In Section 3, we show our numerical analysis. The last section contains our main conclusions.

2. Non-ideal Duffing oscillator

Several mechanical systems can be described by the Duffing equation. Tusset and Balthazar [32] studied ideal and non-ideal Duffing oscillator with chaotic behavior. They suppressed the chaotic oscillations through the application of two control signals. In this work, we consider a non-ideal system consisting of a mass, spring and pendulum. Figure 1 shows a schematic model of the non-ideal oscillator [31], that is composed of a cart (mass M), with a pendulum (mass m and length r), connected to a fixed frame by a nonlinear spring and a dash-pot. We denote by X the displacement of the cart and by φ the angular displacement of the pendulum.

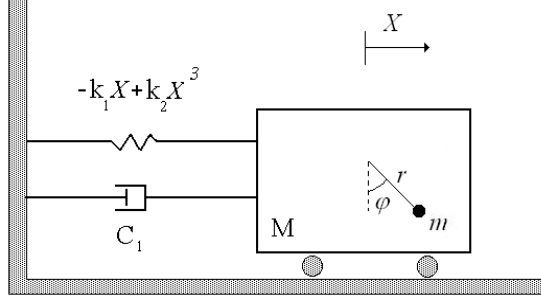


Figure 1: Schematic model of the non-ideal oscillator.

The equations of motion, obtained by using Lagrangian approach, for both the cart and the pendulum are given by:

$$(m + M) \frac{d^2 X}{dt^2} + c_1 \frac{dX}{dt} - k_1 X + k_2 X^3 = mr \left(\frac{d\varphi^2}{dt} \sin \varphi - \frac{d^2 \varphi}{dt^2} \cos \varphi \right), \quad (1)$$

$$mr^2 \frac{d^2 \varphi}{dt^2} + c_2 \frac{d\varphi}{dt} + mgr \sin \varphi = E - mr \frac{d^2 X}{dt^2} \cos \varphi, \quad (2)$$

where E is a constant source of energy. According to Equation (1), for $k_1 < 0$, the Duffing oscillator can be interpreted as a forced oscillator with a spring whose restoring force is $F = k_1 X - k_2 X^3$. Whereas, for $k_1 > 0$, the Duffing oscillator describes the dynamics of a point mass in a double well potential, such as a deflection structure building model.

Considering $x \equiv X/r$ and $\tau \equiv \omega_1 t$ ($\omega_1 \equiv \sqrt{\frac{k_1}{m+M}}$), the equations of motion are rewritten in the following form:

$$\ddot{x} + \beta_1 \dot{x} - x + \gamma x^3 = \varepsilon (\dot{\varphi}^2 \sin \varphi - \ddot{\varphi} \cos \varphi), \quad (3)$$

$$\ddot{\varphi} + \beta_2 \dot{\varphi} + \Omega^2 \sin \varphi = \alpha - \ddot{x} \cos \varphi. \quad (4)$$

for $\beta_1 \equiv \frac{c_1}{(m+M)\omega_1}$, $\gamma \equiv \frac{k_2 r^2}{k_1}$, $\varepsilon \equiv \frac{m}{m+M}$, $\beta_2 \equiv \frac{c_2}{mr^2\omega_1^2}$, $\Omega \equiv \frac{\omega_2}{\omega_1}$ ($\omega_2 \equiv \sqrt{g/r}$), and $\alpha \equiv \frac{E}{mr^2\omega_1^2}$ (source of energy).

These equations of motion correspond to a simplified mathematical model for oscillator with a limited power supply. In this case, the source of energy is given by a DC motor and the parameter α is associated with its input voltage.

3. Arnold tongues and Shrimps

Many systems exhibit behaviour that can be studied by means of two-parameter analysis [33, 34]. Baesens and collaborators [35] studied two-parameter families of torus maps that involve change of mode-locking type. They analysed codimension-1 and -2 bifurcations for flows on the torus, and found a large variety of bifurcation diagrams,

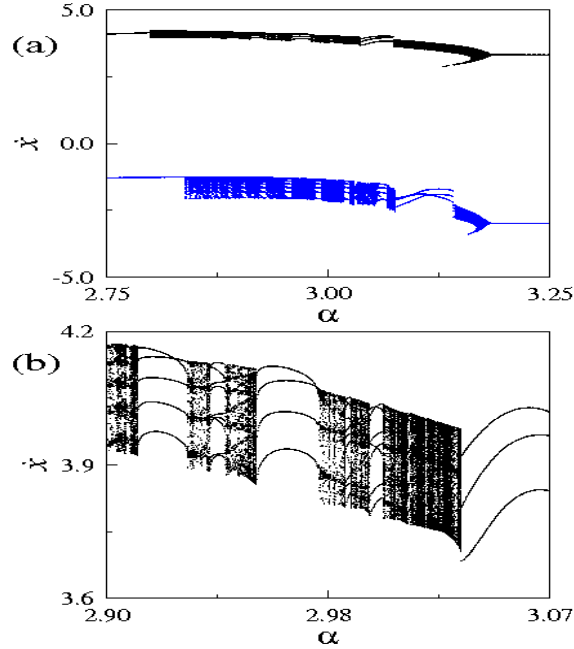


Figure 2: (Color online) (a) Bifurcation diagram showing coexisting attractors for \dot{x} in terms of α with $\varepsilon = 0.09$. (b) Magnification of bifurcation diagram for the attractors plotted in black.

some of them with Arnold tongues. Their numerical studies were focussed on resonance regions revealing a rich collection of codimension-one and -two bifurcations.

We analyze, in this Section, self-similar structures and Arnold tongues in the parameter space of the non-ideal Duffing oscillator were performed by using the fourth-order Runge-Kutta method with a fixed step. The control parameters were fixed at $\beta_1 = 0.05$, $\beta_2 = 1.5$, $\gamma = 0.1$, and $\Omega = 1.0$. We consider for dynamic investigations the variations of parameters ε (the ratio of the masses) and α (input voltage of the DC motor).

First of all, we use a bifurcation diagram, as shown in Figures 2(a) and (b) for $\varepsilon = 0.09$, to verify possible solutions generated by the oscillator. This diagram is constructed varying the control parameter α . For each value of the parameter, we plot the local maximum values of the dynamical variable \dot{x} neglecting the transients. As can be seen in Figures 2(a) and (b), the bifurcation diagram is composed of periodic windows associated with period-adding sequences. See as an example the three main periodic windows in (b), with periodic attractor of periods 5, 4, and 3. Then, as α is increased the period decreases by 1.

In addition, figure 2(a) exhibits the coexistence of two attractors each one plotted with a different color (black and blue). In mechanical systems, the coexistence of attractors is quite common non-linear phenomenon observed. For example, the coexistence of a large number of periodic attractors in a mechanical system was observed by Feudel and collaborators [36]. They studied the kicked double rotor system, and verified the possibility of the system to be stabilized by means of a small perturbation. In an experimental nonlinear pendulum, it was also observed two coexisting attractors [37]. Multiple attrac-

tors may be found in many nonlinear dynamical systems, for instance, driven damped pendulum [39] and spring-pendulum system [40].

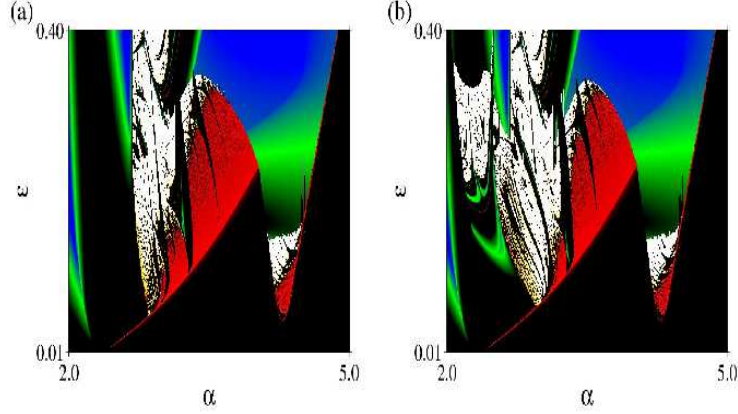


Figure 3: (Color online) Parameter plane diagram for ε versus α with different initial conditions. Periodic solutions are plotted in blue, green and black scale ranges, quasi-periodic in red, and chaotic in yellow and white.

In order to better characterize the dynamics of the oscillator and to examine the structures related to the periodic windows, we construct diagrams of two-dimensional parameter space by using the Lyapunov exponents. To evaluate these exponents, we use the algorithm proposed by Wolf and coworkers [38]. One positive Lyapunov exponent (LLE) indicates a chaotic attractor, all negative exponents a periodic, and two null exponents a quasi-periodic or a bifurcation point.

In Figures 3(a) and (b), we plot the parameter plane diagrams, for ε versus α , using a grid of 800x800 cells. Periodic solutions are plotted in blue, green and black scale ranges, quasi-periodic in red (bifurcation points in red), and chaotic in yellow and white, where the colors, corresponding to range of Lyapunov exponents values, are introduced to emphasize the structures details. To make evident the coexistence of multiple attractors, these diagrams were constructed considering different initial conditions. From an applied perspective, system with coexistence of attractor in noisy environments lead to basin hopping [41, 42], the alternate switching among different attractors.

In Figure 4(a), we provide a magnification of rectangular area of Figure 3(a) revealing many periodic structures (in black) knowing as Arnold tongues [6] that correspond to phase locking, i.e., periodic orbits with the same frequency. Surprisingly, the tongues origins appear for low value of ε for a given α , accumulate in a starting point, namely, the tongues distributions appear highly organized with a codimension-2 bifurcation as a point of accumulations. In this case, the point of accumulations corresponds to a saddle-node Hopf bifurcation [35].

Moreover, we can observe in Figures 4(b) and (c) small periodic structures (blue, green, and black), called shrimps [2], embedded in parameter regions with chaotic regimes (yellow and white). The shrimps are composed of the central body bordered by a saddle-node and a flip bifurcations. As can be noted in Figure 4(d), the shrimp-shaped windows present an interesting feature of self-similar properties. In other words, we can verify in successive magnifications (not shown here) the repetition of these types of windows for

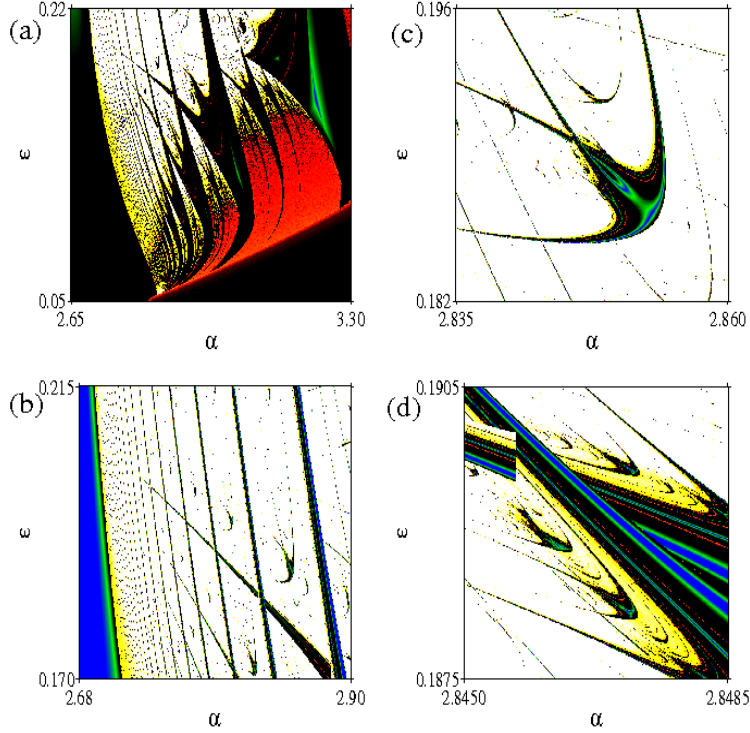


Figure 4: (Color online) (a) Magnification of the parameter plane diagram shown in Fig. 3a. (b)-(d) Successive magnifications. Periodic solutions are plotted in blue, green and black scale ranges, quasi-periodic in red, and chaotic in yellow and white.

different length scales.

In the end, we provide in Figure 5(a) a magnification for a rectangular area involving the accumulation point of the Arnold tongues. The inferior part of tongues present unexpected shape being quite similar to the superior part. In addition, embedded in chaotic regions an uncountable number of shrimps as shown in Figure 5(b).

In addition, the periodic windows in the parameter spaces can be understood as areas delimited by the codimension-2 saddle-node Hopf bifurcation curves. On the other hand, the Arnold tongues appear whenever the ratio between the driven and natural frequencies is a rational number, and they accumulate in a codimension-2 point.

4. Conclusions

We have investigated the parameter space of a non-ideal Duffing oscillator, namely, the mass-spring-pendulum system. Initially, we have identified coexisting attractors with period-adding cascades. In this case, it is possible to verify periodicities of the windows by using the Fibonacci rule [6]. This rule characterizes an integer sequence, in that the sequence is the sum of two previous ones. Moreover, this sequence is related with Golden ration.

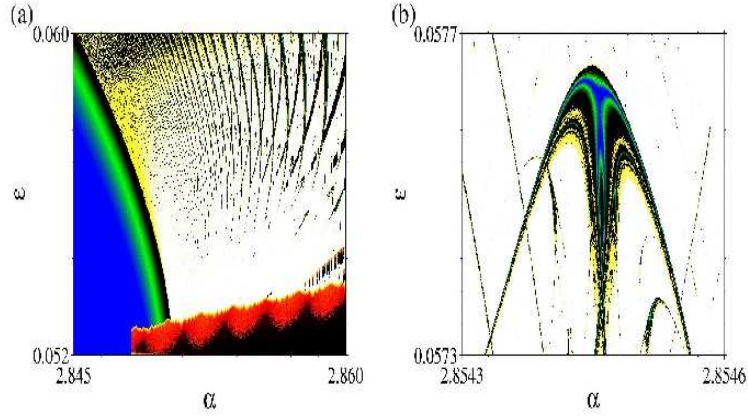


Figure 5: (Color online) (a) Magnification of parameter plane diagram shown in Fig. 3a. (b) Magnification of (a).

When the source of energy was included in the oscillator we were able to observe parameter regions identified as Arnold tongues corresponding to mode locked and periodic motion with a common frequency. The mode locked occurs when the combined motion presented in the mass-spring-pendulum driven by a DC motor becomes periodic. We have verified that locked and unlocked regions were interwoven in parameter space. Our results showed that the tongues origins accumulate in a low value of ε for a given α . Furthermore, the tongues are organized from a codimension-2 bifurcation as a point of accumulations. We also observed shrimp-shaped structures immersed in parameter regions with chaotic regimes, and with highly organized distributions. We have found shrimps in small range of the parameter space.

Moreover, it is important to emphasize that characterization of the attractors in parameter space of applied systems is useful to choose robust periodic orbits and also to evaluate the attractor changes in case of controlling chaotic oscillations.

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